On an optimal quadrature formula in Sobolev space $L_2^{(m)}(0,1)$

Kh.M.Shadimetov, A.R.Hayotov, F.A.Nuraliev

Abstract

In this paper in the space $L_2^{(m)}(0,1)$ the problem of construction of optimal quadrature formulas is considered. Here the quadrature sum consists on values of integrand at nodes and values of first derivative of integrand at the end points of integration interval. The optimal coefficients are found and norm of the error functional is calculated for arbitrary fixed N and for any $m \geq 2$. It is shown that when $m = 2$ and $m = 3$ the Euler-Maclaurin quadrature formula is optimal.

MSC: 65D32.

Keywords: optimal quadrature formulas, error functional, extremal function, Sobolev space, optimal coefficients.

1 Introduction

It is known, that numerical integration formulae, or quadrature formulae, are methods for the approximate evaluation of definite integrals. They are needed for the computation of those integrals for which either the antiderivative of the integrand cannot be expressed in terms of elementary functions or for which the integrand is available only at discrete points, for example from experimental data. In addition and even more important, quadrature formulae provide a basic and important tool for the numerical solution of differential and integral equations.

There are various methods in the theory of quadrature, which allow us approximately calculate integrals with the help of finite number values of integrand. Present work also is devoted to one of such methods, i.e. to construction of optimal quadrature formulas for approximate evaluation of definite integrals in the space $L_2^{(m)}(0,1)$ equipped with the norm

$$\|\varphi(x)\|_{L_2^{(m)}(0,1)} = \left\{ \int_0^1 (\varphi^{(m)}(x))^2 dx \right\}^{1/2}.$$

Consider following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi[\beta] + A\varphi'[0] + B\varphi'[N] \tag{1.1}$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C[\beta]\delta(x-h\beta) + A\delta'(x) + B\delta'(x-1) \tag{1.2}$$

in the space $L_2^{(m)}(0,1)$. Here $C[\beta]$, $\beta=\overline{0,N}$, A and B are the coefficients of the formula (1.1), $[\beta]=h\beta$, $h=\frac{1}{N}$, $N=1,2,...$, $\varepsilon_{[0,1]}(x)$ is the indicator of interval [0,1], $\delta(x)$ is the Dirac delta-function.

The difference

$$(\ell(x),\varphi(x)) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N} C[\beta]\varphi[\beta] - A\varphi'[0] - B\varphi'[N].$$

is called the *error* of the quadrature formula (1.1)

Error of the formula (1.1) is estimated with the help of norm of the error functional (1.2) in the conjugate space $L_2^{(m)*}(0,1)$, i.e. by

$$\left\|\ell(x)|L_2^{(m)*}\right\| = \sup_{\left\|\varphi(x)|L_2^{(m)}\right\|=1} |(\ell(x),\varphi(x))|.$$

Furthermore, norm of the error functional $\ell(x)$ depends on the coefficients $C[\beta]$, A and B. Choice of the coefficients when nodes are fixed is linear problem. Therefore we minimize norm of the functional $\ell(x)$ by coefficients, i.e. we find

$$\left\|\overset{\circ}{\ell}(x)|L_2^{(m)*}\right\| = \inf_{C[\beta],A,B} \left\|\ell(x)|L_2^{(m)*}\right\|. \tag{1.3}$$

If $\left\|\overset{\circ}{\ell}(x)|L_2^{(m)*}\right\|$ is found then the functional $\overset{\circ}{\ell}(x)$ is said to be correspond to the optimal quadrature formula (1.1) in $L_2^{(m)}$ and corresponding coefficients are called *optimal*. Thus we get following problems.

Problem 1. *Find norm of the error functional $\ell(x)$ of quadrature formula of the form (1.1) in the space $L_2^{(m)*}(0,1)$.*

Problem 2. *Find coefficients $C[\beta]$, A and B which satisfy the equality (1.3).*

Problem 2 for quadrature formulas of the form

$$\int_0^N \varphi(x)dx \cong \sum_{k=0}^{N} p_k\varphi(k)$$

on $L_2^{(m)}$ first considered by A.Sard [1]. By A.Sard and S.D.Meyers [2] the solution of this problem was obtained for the following cases: $m = 1$ for arbitrary fixed N; $m = 2$ for $N \leq 20$; $m = 3$ for $N \leq 12$; $m = 4$ for $N \leq 9$.

By I.J.Shoenberg and S.D.Silliman [3] the Sard's problem for $N \to \infty$, i.e. for formula of the form

$$\int\limits_0^\infty \varphi(x)dx \cong \sum_{k=0}^{\infty} B_k^{(m)} \varphi(k)$$

is considered. In [3] an algorithm for finding of the coefficients $B_k^{(m)}$ is given with the help of spline of degree $2m-1$. In the cases $m = 2, 3, ..., 7$ the coefficients $B_k^{(m)}$ are calculated using a Computer. Calculation of these coefficients up to $m = 30$ were done by F.Ya.Zagirova [4].

In [5] in the space $L_2^{(m)}$ considered quadrature formula of the form

$$\int\limits_{-\eta_1}^{N+\eta_2} \omega(x)\varphi(x)dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi[\beta], \tag{1.4}$$

where $0 \leq \eta_j < 1$, $\omega(x)$ is weight function, $C[\beta]$ are the coefficients. In [5] the algorithm for finding optimal coefficients $C[\beta]$ of quadrature formulas of the form (1.4) is given and for the optimal coefficients the system of $2m-2$ linear equations is obtained. These results of S.L.Sobolev generalized above mentioned results of A.Sard, S.D.Meyers, Schoenberg and Silliman. Further realization of Sobolev's algorithm studied by Z.Jamalov, F.Ya.Zagirova, Kh.M.Shadimetov. In [6], [7] the problem of construction of optimal quadrature formulas (1.4) was completely solved for arbitrarily fixed N and for any m in the space $L_2^{(m)}(0, 1)$.

Main goal of the present work is to solve problems 1 and 2 for quadrature formulas of the form (1.1).

2 Definitions and known formulas

In this section we give some definitions and formulas which are necessary in the proofs of main results.

Euler polynomials $E_k(x)$, $k = 1, 2, ...$ is defined by following formula [8]

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2}, \tag{2.1}$$

$E_0(x) = 1$.

For Euler polynomials following identity hold

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right), \tag{2.2}$$

and also following theorem is take placed

Theorem 2.1 [9]. *Polynomial $P_k(x)$ which determined by formula*

$$P_k(x) = (x-1)^{k+1}\sum_{i=0}^{k+1}\frac{\Delta^i 0^{k+1}}{(x-1)^i} \tag{2.3}$$

is the Euler polynomial (2.1) of degree k, i.e. $P_k(x) = E_k(x)$.

Following formula is valid [10]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{2.4}$$

where $\Delta^i\gamma^k$ is finite difference of order i of γ^k, $\Delta^i 0^k = \Delta^i\gamma^k|_{\gamma=0}$.

At last we give following well known formulas from [11]

$$\sum_{\gamma=0}^{\beta-1}\gamma^k = \sum_{j=1}^{k+1}\frac{k!\, B_{k+1-j}}{j!\,(k+1-j)!}\,\beta^j, \tag{2.5}$$

where B_{k+1-j} are Bernoulli numbers,

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu}\left(\begin{array}{c}\nu\\ p\end{array}\right)\Delta^\alpha 0^p x^{\nu-p}. \tag{2.6}$$

3 The extremal function and the representation of the error functional norm

To solve problem 1, i.e. for finding norm of the error functional (1.2) in the space $L_2^{(m)}(0,1)$ concept of the extremal function is used [12]. The function $\psi_\ell(x)$ is said to be *extremal function*

of the error functional (1.2) (see [12]) if following equality holds

$$(\ell(x), \psi_\ell(x)) = \left\| \ell | L_2^{(m)*} \right\| \left\| \psi_\ell | L_2^{(m)} \right\|.$$

In the space $L_2^{(m)}(0,1)$ the extremal function $\psi_\ell(x)$ of the error functional $\ell(x)$ is found by S.L.Sobolev. This extremal function have the form

$$\psi_\ell(x) = (-1)^m \ell(x) * G(x) + P_{m-1}(x), \tag{3.1}$$

where $G(x) = \frac{x^{2m-1} sign x}{2(2m-1)!}$, $P_{m-1}(x)$ is a polynomial of degree $m-1$. Since the functional $\ell(x)$ belongs to the space $L_2^{(m)*}(0,1)$ therefore following holds

$$(\ell(x), x^\alpha) = 0, \qquad \alpha = 0, 1, ..., m-1. \tag{3.2}$$

Norm of the error functional of quadrature formula (1.1) depends on coefficients of this formula. Indeed, since the space $L_2^{(m)}(0,1)$ is Hilbert space, then by using (3.1), taking into account of Riesz theorem about common form of a linear continuous functional on Hilbert space, we get

$$\|\ell\|^2 = (\ell, \psi_\ell) = (-1)^{m+1} \left[\frac{A \cdot B}{(2m-3)!} - 2 \left(A \int_0^1 \frac{x^{2m-2}}{2(2m-2)!}\, dx - B \int_0^1 \frac{(x-1)^{2m-2}}{2(2m-2)!} dx \right) + \right.$$

$$+2 \sum_{\beta=0}^{N} C[\beta] \left(A \frac{(h\beta)^{2m-2}}{2(2m-2)!} - B \frac{(h\beta-1)^{2m-2}}{2(2m-2)!} \right) + 2 \sum_{\beta=0}^{N} C[\beta] \int_0^1 \frac{|x-h\beta|^{2m-1}}{2(2m-1)!} dx -$$

$$\left. - \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C[\beta] C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} - \int_0^1 \int_0^1 \frac{(x-y)^{2m-1} sign(x-y)}{2(2m-1)!} dx dy \right].$$

Thus, the problem 1 is solved for quadrature formulas of the form (1.1) in the space $L_2^{(m)}(0,1)$.

4 The system of Wiener-Hopf type

Now we investigate problem 2. For finding of minimum of the $\|\ell\|^2$ under the conditions (3.2) Lagrange method of undetermined multipliers is used. For this we consider following function

$$\Psi = \|\ell\|^2 + 2 \cdot (-1)^{m+1} \sum_{\alpha=0}^{m-1} \lambda_\alpha (\ell, x^\alpha).$$

Equating to zero partial derivatives by coefficients $C[\beta]$, A and B, together with conditions (3.2) we get following system of linear equations

$$\sum_{\gamma=0}^{N} C[\gamma]\frac{|h\beta-h\gamma|^{2m-1}}{2(2m-1)!}-A\frac{(h\beta)^{2m-2}}{2(2m-2)!}+$$

$$+B\frac{(h\beta-1)^{2m-2}}{2(2m-2)!}+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha=\int\limits_0^1\frac{|x-h\beta|^{2m-1}}{2(2m-1)!}dx,\quad \beta=\overline{0,N}, \tag{4.1}$$

$$\sum_{\gamma=0}^{N} C[\gamma]\frac{(h\gamma)^{2m-2}}{2(2m-2)!}+\frac{B}{2(2m-3)!}-\lambda_1=\frac{1}{2(2m-1)!}, \tag{4.2}$$

$$\sum_{\gamma=0}^{N} C[\gamma]\frac{(h\gamma-1)^{2m-2}}{2(2m-2)!}-\frac{A}{2(2m-3)!}+\sum_{\alpha=1}^{m-1}\alpha\lambda_\alpha=\frac{1}{2(2m-1)!}, \tag{4.3}$$

$$\sum_{\gamma=0}^{N} C[\gamma]=1, \tag{4.4}$$

$$\sum_{\gamma=0}^{N} C[\gamma]h\gamma+A+B=\frac{1}{2}, \tag{4.5}$$

$$\sum_{\gamma=0}^{N} C[\gamma](h\gamma)^\alpha+\alpha B=\frac{1}{\alpha+1},\quad \alpha=\overline{2,m-1}. \tag{4.6}$$

The system (4.1)-(4.6) is called by *system of Wiener-Hof type* for the optimal coefficients [12]. In the system (4.1)-(4.6) coefficients $C[\beta]$, $\beta=\overline{0,N}$, A and B, and also λ_α, $\alpha=\overline{0,m-1}$ are unknowns. The system (4.1)-(4.6) has unique solution. The proof of existence and uniqueness of the solution of this system is as the proof of existence and uniqueness of the solution of Wiener-Hopf system of the optimal coefficients in the space $L_2^{(m)}(0,1)$ for quadrature formulas of the form (1.4) (see [13]).

5 The optimal coefficients and norm of the error functional

In present section we study solution of the system (4.1)-(4.6). In the solution of this system we use the approach which used in solution of the linear system for optimal coefficients of quadrature formulas of the form (1.4) in [6].

5.1 The coefficients of optimal quadrature formulas

It is easy to prove following lemma for the coefficients $C[\beta]$ of quadrature formulas of the form (1.1).

Lemma 5.1. *The optimal coefficients $C[\beta]$, $1 \leq \beta \leq N-1$, of quadrature formulas of the form (1.1) have following form*

$$C[\beta] = h\left(1 + \sum_{k=1}^{m-1}\left(d_k q_k^{\beta} + p_k q_k^{N-\beta}\right)\right), \qquad 1 \leq \beta \leq N-1, \tag{5.1}$$

where d_k, p_k are unknowns, q_k are roots of the Euler polynomial $E_{2m-2}(q)$, $|q_k| < 1$.

Lemma is proved as lemma 3 of the work [9] and in the proof the discrete analogue $D_m[\beta]$ of the polyharmonic operator $\frac{d^{2m}}{dx^{2m}}$ is used. The discrete analogue $D_m[\beta]$ of the polyharmonic operator $\frac{d^{2m}}{dx^{2m}}$ is constructed in [14].

We need following lemmas in proof of main results.

Lemma 5.2. *Following identity is take placed*

$$\sum_{i=0}^{\alpha} \frac{dq + pq^{N+i}(-1)^{i+1}}{(q-1)^{i+1}} \Delta^i 0^{\alpha} = (-1)^{\alpha+1} \sum_{i=0}^{\alpha} \frac{dq^i + pq^{N+1}(-1)^{i+1}}{(1-q)^{i+1}} \Delta^i 0^{\alpha}, \tag{5.2}$$

here α and N are natural numbers, $\Delta^i 0^{\alpha}$ is finite difference of order i of γ^{α} at the point 0.

Proof. For convenience left and right sides of (5.2)we denote by L_1 L_2 respectively, i.e.

$$L_1 = \sum_{i=0}^{\alpha} \frac{dq + pq^{N+i}(-1)^{i+1}}{(q-1)^{i+1}} \Delta^i 0^{\alpha} \quad L_2 = (-1)^{\alpha+1} \sum_{i=0}^{\alpha} \frac{dq^i + pq^{N+1}(-1)^{i+1}}{(1-q)^{i+1}} \Delta^i 0^{\alpha}.$$

First consider L_1. By using the equality (2.3) and identity (2.2) for L_1 consequently we get

$$L_1 = \sum_{i=0}^{\alpha} \frac{dq + pq^{N+i}(-1)^{i+1}}{(q-1)^{i+1}} \Delta^i 0^{\alpha} = \frac{dq}{(q-1)^{\alpha+1}} E_{\alpha-1}(q) + \frac{pq^{N+\alpha}(-1)^{\alpha+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}\left(\frac{1}{q}\right) =$$

$$= \frac{dq}{(q-1)^{\alpha+1}} E_{\alpha-1}(q) + \frac{pq^{N+\alpha}(-1)^{\alpha+1}}{(q-1)^{\alpha+1}} \frac{E_{\alpha-1}(q)}{q^{\alpha-1}} = \frac{dq + pq^{N+1}(-1)^{\alpha+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}(q). \tag{5.3}$$

Similarly for L_2 by using (2.3) and (2.2) we have

$$L_2 = \sum_{i=0}^{\alpha} \frac{dq^i + pq^{N+1}(-1)^{i+1}}{(1-q)^{i+1}} \Delta^i 0^{\alpha} = \frac{dq^{\alpha}}{(q-1)^{\alpha+1}} E_{\alpha-1}\left(\frac{1}{q}\right) + \frac{pq^{N+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}(q) =$$

$$= \frac{dq}{(1-q)^{\alpha+1}} E_{\alpha-1}(q) + \frac{pq^{N+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}(q) = \frac{dq(-1)^{\alpha+1} + pq^{N+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}(q) =$$

$$= (-1)^{\alpha+1} \frac{dq + pq^{N+1}(-1)^{\alpha+1}}{(q-1)^{\alpha+1}} E_{\alpha-1}(q). \tag{5.4}$$

From (5.3) and (5.4) clear, that $L_1 = (-1)^{\alpha+1} L_2$. Lemma 5.2 is proved.

We denote

$$Z_p = \sum_{k=1}^{m-1} \sum_{i=0}^{p} \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1-q_k)^{i+1}} \Delta^i 0^p. \tag{5.4*}$$

Lemma 5.3. *Following identities are valid*

$$\sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{(j-1)!} \sum_{i=1}^{2m-3-j} \frac{B_{2m-j-i} h^{2m-j-i}}{i!\ (2m-j-i)!} =$$

$$= \sum_{j=3}^{m} \frac{B_j h^j}{j!} \sum_{i=0}^{m-2} \frac{(-1)^i}{i!\ (2m-1-j-i)!} + \sum_{j=m+1}^{2m-2} \frac{B_j h^j}{j!} \sum_{i=0}^{2m-2-j} \frac{(-1)^i}{i!\ (2m-1-j-i)!}$$

and

$$\sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{(j-1)!} \sum_{p=2}^{2m-1-j} \frac{h^{p+1} Z_p}{p!\ (2m-1-j-p)!} =$$

$$= \sum_{j=3}^{m+1} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{l=0}^{m-2} \frac{(-1)^l}{l!\ (2m-1-j-l)!} + \sum_{j=m+2}^{2m-1} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{l=0}^{2m-1-j} \frac{(-1)^l}{l!\ (2m-1-j-l)!}.$$

The proof of lemma is obtained by expansion in powers of h of left sides of given identities.

For the coefficients of optimal quadrature formulas of the form (1.1) following theorem holds.

Theorem 5.1. *Among quadrature formulas of the form (1.1) with the error functional (1.2) there exists unique optimal formula which coefficients are determined by following formulas*

$$C[0] = h\left(\frac{1}{2} + \sum_{k=1}^{m-1} \frac{p_k q_k^N - d_k q_k}{1-q_k}\right), \tag{5.5}$$

$$C[\beta] = h\left(1 + \sum_{k=1}^{m-1} (d_k q_k^\beta + p_k q_k^{N-\beta})\right), \qquad \beta = \overline{1, N-1}, \tag{5.6}$$

$$C[N] = h\left(\frac{1}{2} + \sum_{k=1}^{m-1} \frac{d_k q_k^N - p_k q_k}{1-q_k}\right), \tag{5.7}$$

$$A = h^2\left(\frac{1}{12} - \sum_{k=1}^{m-1} \frac{d_k q_k + p_k q_k^{N+1}}{(1-q_k)^2}\right), \tag{5.8}$$

$$B = h^2 \left(-\frac{1}{12} + \sum_{k=1}^{m-1} \frac{d_k q_k^{N+1} + p_k q_k}{(1-q_k)^2} \right), \tag{5.9}$$

where d_k and p_k satisfy following system $2m-2$ linear equations:

$$\sum_{k=1}^{m-1} \sum_{i=0}^{j} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^j = \frac{B_{j+1}}{j+1}, \quad j = \overline{2, m-1}, \tag{5.10}$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2m-2} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-2} = 0, \tag{5.11}$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{j} (1 - q_k^N) \frac{(-1)^{i+1} d_k q_k^i - p_k q_k}{(q_k - 1)^{i+1}} \Delta^i 0^j = 0, \quad j = \overline{2, m-1}. \tag{5.12}$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2m-2} \frac{(-1)^{i+1} d_k q_k^{N+i} + p_k q_k}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-2} = 0. \tag{5.13}$$

Here B_α are Bernoulli numbers, $\Delta^i \gamma^j$ is difference of order i of γ^j, $\Delta^i 0^j = \Delta^i \gamma^j|_{\gamma=0}$, q_k are roots of Euler polynomial of degree $2m-2$, $|q_k| < 1$.

Proof. First we give plan of proof.

From (5.1) clear that instead of unknowns $C[\beta]$, $\beta = \overline{1, N-1}$ it is sufficient to find unknowns d_k, p_k, $k = \overline{1, m-1}$. The coefficients $C[0]$, $C[N]$, A, B and λ_α, $\alpha = \overline{0, m-1}$ are expressed by d_k and p_k, $k = \overline{1, m-1}$. So if we find d_k and p_k, then the system (4.1)-(4.6) is solved completely. Substituting the equality (5.1) to equation (4.1) we get polynomial of degree $2m$ of $h\beta$ on both sides of (4.1). Equating coefficients of same degrees of $h\beta$ we find λ_α, $\alpha = \overline{0, m-1}$, $C[0]$, A and system (5.10) for d_k, p_k. Taking account of (5.1), (5.5), (5.9), from conditions (4.4) and (4.5) we get (5.7), (5.9), i.e. we obtain $C[N]$ and B. Further, by using (5.1), (5.9) and expression for λ_1, from (4.2) we get the equation (5.11). System of equations (5.12) for unknowns d_k ,p_k , we obtain from equation (4.6), using (5.1), (5.5)-(5.9). Finally, taking into account (5.1), (5.8) and λ_α, $\alpha = \overline{1, m-1}$ from equation (4.3) we have (5.13).

Further we give detailed explanation of proof of the theorem.

First we consider first sum of equation (4.1). For this sum we have

$$S = \sum_{\gamma=0}^{N} C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} =$$

$$= C[0]\frac{(h\beta)^{2m-1}}{(2m-1)!} + \sum_{\gamma=1}^{\beta} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!} - \sum_{\gamma=0}^{N} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}.$$

Lat two sums of the expression S we denote

$$S_1 = \sum_{\gamma=1}^{\beta} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!}, \quad S_2 = \sum_{\gamma=0}^{N} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}$$

and we calculate them separately.

By using lemma 5.1 and formulas (2.4), (2.5) for S_1 we have

$$S_1 = \sum_{\gamma=0}^{\beta} h\left(1 + \sum_{k=1}^{m-1}\left(d_k q_k^{\gamma} + p_k q_k^{N-\gamma}\right)\right)\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!} =$$

$$= \frac{h^{2m}}{(2m-1)!}\left[\sum_{\gamma=0}^{\beta-1}\gamma^{2m-1} + \sum_{k=1}^{m-1}\left(d_k q_k^{\beta}\sum_{\gamma=0}^{\beta-1} q_k^{-\gamma}\gamma^{2m-1} + p_k q_k^{N-\beta}\sum_{\gamma=0}^{\beta-1} q_k^{\gamma}\gamma^{2m-1}\right)\right] =$$

$$= \frac{h^{2m}}{(2m-1)!}\left[\sum_{j=1}^{2m}\frac{(2m-1)!B_{2m-j}}{j!\cdot(2m-j)!}\beta^j + \sum_{k=1}^{m-1}\left[d_k q_k^{\beta}\left\{\frac{q_k}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i 0^{2m-1}}{(q_k-1)^i} -\right.\right.\right.$$

$$\left. - \frac{q_k^{1-\beta}}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i\beta^{2m-1}}{(q_k-1)^i}\right\} + p_k q_k^{N-\beta}\left\{\frac{1}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{q_k-1}\right)^i\Delta^i 0^{2m-1} -\right.$$

$$\left.\left.\left. - \frac{q_k^{\beta}}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{q_k-1}\right)^i\Delta^i\beta^{2m-1}\right\}\right]\right].$$

Taking into account that q_k is a root of Eulaer polynomial $E_{2m-2}(q)$ an using formulas (2.3), (2.6) the expression for S_1 we reduce to following form

$$S_1 = \frac{(h\beta)^{2m}}{(2m)!} + h\cdot\frac{(h\beta)^{2m-1}}{(2m-1)!}B_1 + h^{2m}\sum_{j=1}^{2m-2}\frac{B_{2m-j}}{j!(2m-j)!}\beta^j +$$

$$+ h^{2m}\sum_{j=0}^{2m-1}\frac{\beta^{2m-1-j}}{j!(2m-1-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{-d_k q_k + p_k q_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i 0^j. \tag{5.14}$$

Now consider S_2. By using conditions of orthogonality (4.4)-(4.6) the expression S_2 we rewrite by powers of $h\beta$

$$S_2 = \sum_{\gamma=0}^{N} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!} = \frac{1}{2}\sum_{j=2}^{m-1}\frac{(h\beta)^{2m-1-j}}{j!(2m-1-j)!}\left(\frac{1}{j+1} - jB\right) -$$

$$-\frac{(h\beta)^{2m-2}}{2(2m-2)!}\left(\frac{1}{2}-A-B\right)+\frac{(h\beta)^{2m-1}}{2(2m-1)!}+\frac{1}{2}\sum_{j=m}^{2m-1}\frac{(h\beta)^{2m-1-j}}{j!(2m-1-j)!}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^j. \quad (5.15)$$

The right side of the equation (4.1) have following form

$$\int_0^1 \frac{|x-h\beta|^{2m-1}}{2(2m-1)!}dx=\frac{(h\beta)^{2m}}{(2m)!}+\sum_{j=0}^{2m-1}\frac{(-h\beta)^{2m-1-j}}{2(2m-1-j)!(j+1)!} \quad (5.16)$$

Substituting (5.16) and S into equation (4.1) and using (5.14), (5.15) we have

$$\frac{(h\beta)^{2m}}{(2m)!}+C[0]\frac{(h\beta)^{2m-1}}{(2m-1)!}+h\frac{(h\beta)^{2m-1}}{(2m-1)!}B_1+\sum_{j=1}^{2m-2}\frac{B_{2m-j}h^{2m-j}(h\beta)^j}{j!(2m-j)!}+$$

$$+\sum_{j=0}^{2m-1}\frac{h^{j+1}(h\beta)^{2m-1-j}}{j!(2m-1-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{-d_kq_k+p_kq_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i 0^j-$$

$$-\frac{1}{2}\sum_{j=2}^{m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{j!(2m-1-j)!}\left(\frac{1}{j+1}-jB\right)+\frac{(h\beta)^{2m-2}}{2\ (2m-2)!}\left(\frac{1}{2}-A-B\right)-$$

$$-\frac{(h\beta)^{2m-1}}{2(2m-1)!}-\frac{1}{2}\sum_{j=m}^{2m-1}\frac{(h\beta)^{2m-1-j}}{j!(2m-1-j)!}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^j-A\frac{(h\beta)^{2m-2}}{2(2m-2)!}+$$

$$+B\sum_{j=0}^{2m-2}\frac{(h\beta)^{2m-2-j}(-1)^j}{2\cdot j!\cdot(2m-2-j)!}+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha=\frac{(h\beta)^{2m}}{(2m)!}+\sum_{j=0}^{2m-1}\frac{(-h\beta)^{2m-1-j}}{2\cdot(2m-1-j)!\cdot(j+1)!}.$$

Hence equating coefficients of same powers of $h\beta$ gives

$$\lambda_j=\frac{1}{(2m-1-j)!\cdot j!}\left(\frac{(-1)^{2m-j}}{2(2m-j)}-\frac{B_{2m-j}h^{2m-j}}{2m-j}-\right.$$

$$-h^{2m-j}\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{-d_kq_k+p_kq_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i 0^{2m-1-j}+$$

$$\left.+\frac{1}{2}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^{2m-1-j}-(2m-1-j)\cdot B\cdot\frac{(-1)^{2m-2-j}}{2}\right),\quad j=1,2,...,m-1, \quad (5.17)$$

$$\lambda_0=\frac{1}{2\cdot(2m)!}+\frac{1}{2\cdot(2m-1)!}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^{2m-1}-B\frac{1}{2\cdot(2m-2)!}, \quad (5.18)$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i 0^j=\frac{B_{j+1}}{j+1},\quad j=\overline{2,m-1}, \quad (5.19)$$

$$C[0]=h\left(\frac{1}{2}+\sum_{k=1}^{m-1}\frac{p_kq_k^N-d_kq_k}{1-q_k}\right), \tag{5.20}$$

$$A=h^2\left(\frac{1}{12}-\sum_{k=1}^{m-1}\frac{d_kq_k+p_kq_k^{N+1}}{(1-q_k)^2}\right). \tag{5.21}$$

Here equation (5.19) is the equation (5.10) for unknowns d_k and p_k.

Substituting expressions (5.20) and (5.21) into (4.4) and (4.5), also taking into account (5.1), we find $C[N]$ and B, which have following form

$$C[N]=h\left(\frac{1}{2}+\sum_{k=1}^{m-1}\frac{d_kq_k^N-p_kq_k}{1-q_k}\right), \tag{5.22}$$

$$B=h^2\left(-\frac{1}{12}+\sum_{k=1}^{m-1}\frac{d_kq_k^{N+1}+p_kq_k}{(1-q_k)^2}\right). \tag{5.23}$$

Now substituting the expression of λ_1 from (5.17) when $j=1$ into (4.2), we get one more equation with respect to unknowns d_k and p_k:

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2m-2}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2m-2}=0, \tag{5.24}$$

i.e. we obtain the equation (5.11).

Next, to obtain (5.12) we use equation (4.6). Consider equation (4.6). Since $\alpha=\overline{2,m-1}$, then

$$\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^\alpha+\alpha\,B=\sum_{\gamma=1}^{N-1}C[\gamma](h\gamma)^\alpha+C[N]+\alpha\,B=\frac{1}{\alpha+1}. \tag{5.25}$$

We denote $L=\sum\limits_{\gamma=1}^{N-1}C[\gamma](h\gamma)^\alpha$. Using (5.1), (2.4), (2.5) for L we get

$$L=h^{\alpha+1}\left(\sum_{\gamma=1}^{N-1}\gamma^\alpha+\sum_{k=1}^{m-1}\left(d_k\sum_{\gamma=1}^{N-1}q_k^\gamma\gamma^\alpha+p_kq_k^N\sum_{\gamma=1}^{N-1}q_k^{-\gamma}\gamma^\alpha\right)\right)=$$

$$=\sum_{j=1}^{\alpha+1}\frac{\alpha!B_{\alpha+1-j}}{j!\,(\alpha+1-j)!}h^{\alpha+1-j}+h^{\alpha+1}\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^\alpha-$$

$$-h^{\alpha+1}\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^iN^\alpha.$$

Hence taking into account(2.6) and grouping in powers of h, we have

$$L=\frac{1}{\alpha+1}+\sum_{j=1}^{\alpha}\frac{\alpha!h^j}{(j-1)!\ (\alpha+1-j)!}\left(\frac{B_j}{j}-\sum_{k=1}^{m-1}\sum_{i=0}^{j-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j-1}\right)+$$

$$+h^{\alpha+1}\left(\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^{i}+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{\alpha}-\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{\alpha}\right).$$

Substitution of obtained expression of L to (5.25) gives

$$\frac{1}{\alpha+1}+\sum_{j=1}^{\alpha}\frac{\alpha!h^j}{(j-1)!\ (\alpha+1-j)!}\left(\frac{B_j}{j}-\sum_{k=1}^{m-1}\sum_{i=0}^{j-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j-1}\right)+$$

$$+h^{\alpha+1}\left(\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}(1-q_k^N)\frac{(-1)^{i+1}d_kq_k^i-p_kq_k}{(q_k-1)^{i+1}}\Delta^i0^{\alpha}\right)+C[N]+\alpha B=\frac{1}{\alpha+1}.$$

Hence keeping in mind (5.22), (5.23) we get

$$\sum_{j=3}^{\alpha}\frac{\alpha!h^j}{(j-1)!\ (\alpha+1-j)!}\left(\frac{B_j}{j}-\sum_{k=1}^{m-1}\sum_{i=0}^{j-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j-1}\right)+$$

$$+h^{\alpha+1}\left(\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}(1-q_k^N)\frac{(-1)^{i+1}d_kq_k^i-p_kq_k}{(q_k-1)^{i+1}}\Delta^i0^{\alpha}\right)=0. \tag{5.26}$$

Clearly that the left side of (5.26) is polynomial of degree $\alpha+1$ with respect to h. From (5.26) we obtain that each coefficient of this polynomial is equal to zero, i.e.

$$\sum_{k=1}^{m-1}\sum_{i=0}^{j-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j-1}=\frac{B_j}{j},\quad j=\overline{3,\alpha} \tag{5.27}$$

and

$$\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}(1-q_k^N)\frac{(-1)^{i+1}d_kq_k^i-p_kq_k}{(q_k-1)^{i+1}}\Delta^i0^{\alpha}=0. \tag{5.28}$$

Since $\alpha=\overline{2,m-1}$ then from (5.27) and (5.28) we have

$$\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j}=\frac{B_{j+1}}{j+1},\quad j=\overline{2,m-2} \tag{5.29}$$

and

$$\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^{i}+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{\alpha}=\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{\alpha},\quad \alpha=\overline{2,m-1}. \tag{5.30}$$

Hence by using (5.30) and lemma 5.2 it is easy to show the system of equations (5.29) is the part of (5.19). Thus here we get only system of equations (5.30) which is (5.12).

Now consider the last equation, i.e. the equation (4.3). Difference of the left and the right sides of equation (4.3) we denote by K, i.e.

$$K=\sum_{\gamma=0}^{N} C[\gamma]\frac{(h\gamma-1)^{2m-2}}{2(2m-2)!}-\frac{A}{2(2m-3)!}+\sum_{\alpha=1}^{m-1}\alpha\lambda_\alpha-\frac{1}{2(2m-1)!}$$

and therefore we keep in mind that $K=0$.

Applying binomial formula for first expression of K and taking into account (4.4)-(4.6), after some simplifications

$$\begin{aligned}K=&-\frac{1}{2(2m-1)!}+\frac{1}{2(2m-2)!}-\frac{1}{2(2m-3)!}\left(\frac{1}{2}-B\right)+\sum_{j=1}^{2m-4}\frac{(-1)^j}{2j!(2m-1-j)!}+\\&+\sum_{j=1}^{m-1}\frac{(-1)^j}{2(2m-j)!(j-1)!}-\sum_{j=m-1}^{2m-4}\frac{(-1)^jB}{2j!(2m-3-j)!}-\\&-\sum_{j=1}^{m-1}\frac{(-1)^jB}{2(2m-2-j)!(j-1)!}+\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!(2m-1-j)!}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^{2m-1-j}+\\&+\sum_{j=1}^{m-1}\frac{h^{2m-j}}{(2m-1-j)!\,(j-1)!}\left[\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2m-1-j}-\frac{B_{2m-j}}{2m-j}\right].\end{aligned}\tag{5.31}$$

Now consider the sum $\sum\limits_{\gamma=0}^{N} C[\gamma](h\gamma)^{2m-1-j}$ in (5.31). For this sum using formulas (5.1), (2.4)-(2.6) after some simplifications we obtain

$$\begin{aligned}\sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{2m-1-j}=&\sum_{i=1}^{2m-j}\frac{(2m-1-j)!\,B_{2m-j-i}}{i!\,(2m-j-i)!}h^{2m-j-i}+\\&+h^{2m-j}\left(\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-1-j}-\right.\\&\left.-\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-1-j}\right)-\\&-\sum_{p=0}^{2m-2-j}\frac{(2m-1-j)!\,h^{p+1}}{p!\,(2m-1-j-p)!}\sum_{k=1}^{m-1}\sum_{i=0}^{p}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^p.\end{aligned}\tag{5.32}$$

Substituting (5.32) into (5.31) we get polynomial of degree $2m-1$ with respect to h. It is easy to see that constant term and coefficients in front of h and h^2 are zero. Then for K we obtain

$$K=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!(2m-1-j)!}\left(\sum_{i=1}^{2m-3-j}\frac{(2m-1-j)!B_{2m-j-i}}{i!\,(2m-j-i)!}h^{2m-j-i}-\right.$$
$$\left.-\sum_{p=2}^{2m-2-j}\frac{(2m-1-j)!h^{p+1}}{p!\,(2m-1-j-p)!}\sum_{k=1}^{m-1}\sum_{i=0}^{p}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^p\right)+$$
$$+\sum_{j=1}^{m-1}\frac{h^{2m-j}}{(j-1)!(2m-1-j)!}\left((-1)^{j-1}\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-1-j}+\right.$$
$$+(-1)^j\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-1-j}+$$
$$\left.+\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-1-j}\right)-\sum_{j=1}^{m-1}\frac{h^{2m-j}B_{2m-j}}{(2m-j)!(j-1)!}.$$

Hence if take into account lemma 5.2 and (5.4*), then the expression in the second parenthesis are simplified and K has the form

$$K=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!}\sum_{i=1}^{2m-3-j}\frac{B_{2m-j-i}h^{2m-j-i}}{i!\,(2m-j-i)!}-\sum_{j=1}^{m-1}\frac{h^{2m-j}B_{2m-j}}{(2m-j)!\,(j-1)!}-$$
$$-\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!}\sum_{p=2}^{2m-1-j}\frac{h^{p+1}}{p!\,(2m-1-j-p)!}Z_p. \tag{5.33}$$

Using lemma 5.3 from (5.33) we get

$$K=\sum_{j=3}^{m}\frac{B_jh^j}{j!}\sum_{i=0}^{m-2}\frac{(-1)^i}{i!\,(2m-1-j-i)!}+\sum_{j=m+1}^{2m-2}\frac{B_jh^j}{j!}\sum_{i=0}^{2m-2-j}\frac{(-1)^i}{i!\,(2m-1-j-i)!}-$$
$$-\sum_{j=1}^{m-1}\frac{h^{2m-j}B_{2m-j}}{(2m-j)!\,(j-1)!}-\sum_{j=3}^{m+1}\frac{h^jZ_{j-1}}{(j-1)!}\sum_{i=0}^{m-2}\frac{(-1)^i}{i!\,(2m-1-j-i)!}-$$
$$-\sum_{j=m+2}^{2m-1}\frac{h^jZ_{j-1}}{(j-1)!}\sum_{i=0}^{2m-1-j}\frac{(-1)^i}{i!\,(2m-1-j-i)!}=0.$$

Hence using (5.19) after simplifications we have

$$K=\frac{h^m}{(m-1)!}\left(\frac{B_m}{m}-Z_{m-1}\right)\sum_{i=0}^{m-2}\frac{(-1)^i}{i!\,(m-1-i)!}+$$

$$+\sum_{j=m+1}^{2m-2}\frac{B_jh^j}{j!}\sum_{i=1}^{2m-2-j}\frac{(-1)^i}{i!(2m-1-j-i)!}-$$

$$-\sum_{j=m+2}^{2m-2}\frac{h^jZ_{j-1}}{(j-1)!(2m-1-j)!}\sum_{i=0}^{2m-1-j}\frac{(2m-1-j)!(-1)^i}{i!\,(2m-1-j-i)!}-\frac{h^{2m-1}Z_{2m-2}}{(2m-2)!}=$$

$$=\frac{h^m(-1)^m}{[(m-1)!]^2}\left(\frac{B_m}{m}-Z_{m-1}\right)+\sum_{j=m+1}^{2m-2}\frac{B_jh^j}{j!\,(2m-1-j)!}\left((-1)^j-1\right)-\frac{h^{2m-1}Z_{2m-2}}{(2m-2)!}.$$

Here the middle sum is equal to zero because when j is even $(-1)^j-1=0$, and when j is odd Bernoulli numbers $B_j=0$. Therefore finally for K we get

$$K=-\frac{h^{2m-1}Z_{2m-2}}{(2m-2)!}+\frac{h^m(-1)^m}{[(m-1)!]^2}\left(\frac{B_m}{m}-Z_{m-1}\right)=0.$$

It means hence taking into account (5.4*) we get following equation

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{m-1}=\frac{B_m}{m},\tag{5.34}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2m-2}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-2}=0.\tag{5.35}$$

Using (5.30) when $\alpha=1$ and taking into account lemma 5.2 it is easy to show , that the equation (5.34) coincide by equation of the system (5.19) when $\alpha=1$. Thus, we have obtained the last equation (5.35) for d_k and p_k, which is the same as (5.13). Theorem is proved.

5.2 Calculation of norm of the error functional

For square of norm of the error functional (1.2) of optimal quadrature formulas of the form (1.1) take placed following theorem.

Theorem 5.2. *For square of norm of the error functional (1.2) of optimal quadrature formula of the form (1.1) following is valid*

$$\left\|\ell|L_2^{(m)*}(0,1)\right\|^2=(-1)^{m+1}\left[\frac{B_{2m}h^{2m}}{(2m)!}+\frac{h^{2m+1}}{(2m)!}\sum_{k=1}^{m-1}\sum_{i=0}^{2m}\frac{(1-q_k^N)(d_kq_k^i+(-1)^ip_kq_k)}{(1-q_k)^{i+1}}\Delta^i0^{2m}\right],$$

where d_k, p_k are determined from system (5.10)-(5.13), B_{2m} are Bernoulli numbers, q_k are roots of Euler polynomial of degree $2m-2$, $|q_k|<1$.

Proof. Computing defined integrals in the expression $||\ell||^2$ we get

$$||\ell||^2=(-1)^{m+1}\left[\frac{A\cdot B}{(2m-3)!}-\frac{A-B}{(2m-1)!}+\sum_{\beta=0}^{N}C[\beta]\,\frac{A\,(h\beta)^{2m-2}-B\,(h\beta-1)^{2m-2}}{2(2m-2)!}+\right.$$

$$+\sum_{\beta=0}^{N}C[\beta]\,F(h\beta)+\sum_{\beta=0}^{N}C[\beta]\left\{F(h\beta)-\sum_{\gamma=0}^{N}C[\gamma]\frac{|h\beta-h\gamma|^{2m-1}}{2(2m-1)!}+\right.$$

$$\left.\left.+\frac{A\,(h\beta)^{2m-2}-B\,(h\beta-1)^{2m-2}}{2(2m-2)!}\right\}-\frac{1}{(2m+1)!}\right],$$

where $F(h\beta)$ is determine by formula (5.16). As is obvious from here according to (4.1) the expression into curly brackets is equal to the polynomial $P_{m-1}(h\beta)=\sum\limits_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha$. Then $||\ell||^2$ have the form

$$||\ell||^2=(-1)^{m+1}\left[\frac{A\cdot B}{(2m-3)!}-\frac{A-B}{(2m-1)!}+\sum_{\beta=0}^{N}C[\beta]\,\frac{A\,(h\beta)^{2m-2}-B\,(h\beta-1)^{2m-2}}{2(2m-2)!}+\right.$$

$$\left.+\sum_{\beta=0}^{N}C[\beta]\,F(h\beta)+\sum_{\beta=0}^{N}C[\beta]\;P_{m-1}(h\beta)-\frac{1}{(2m+1)!}\right].$$

Hence using (4.2) and (4.3) we get

$$||\ell||^2=(-1)^{m+1}\left[\frac{A\cdot B}{(2m-3)!}-\frac{A-B}{(2m-1)!}+A\cdot\left(\frac{1}{2(2m-1)!}+\lambda_1-\frac{B}{2(2m-3)!}\right)-\right.$$

$$-B\cdot\left(\frac{1}{2(2m-1)!}-\sum_{\alpha=1}^{m-1}\alpha\lambda_\alpha+\frac{A}{2(2m-3)!}\right)+$$

$$\left.+\sum_{\beta=0}^{N}C[\beta]\,F(h\beta)+\sum_{\beta=0}^{N}C[\beta]\;P_{m-1}(h\beta)-\frac{1}{(2m+1)!}\right]. \tag{5.35}$$

From (5.35) after simplifications using (5.17), (5.12) we have

$$||\ell||^2=(-1)^{m+1}\left[\frac{B-A}{2(2m-1)!}+\frac{A}{(2m-2)!}\cdot\left(-\frac{1}{2(2m-1)}+\frac{1}{2}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^{2m-2}+\right.\right.$$

$$\left.+\frac{(2m-2)B}{2}\right)+B\cdot\sum_{j=1}^{m-1}\frac{1}{(2m-1-j)!(j-1)!}\left(\frac{(-1)^{2m-j}}{2(2m-j)}-\frac{B_{2m-j}h^{2m-j}}{2m-j}-\right.$$

$$-h^{2m-j}\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{-d_kq_k+p_kq_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i0^{2m-1-j}+\frac{1}{2}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^{2m-1-j}-$$

$$-(2m-1-j)B\frac{(-1)^j}{2}\Bigg) + \sum_{\beta=0}^{N} C[\beta]\, F(h\beta) + \sum_{\beta=0}^{N} C[\beta]\, P_{m-1}(h\beta) - \frac{1}{(2m+1)!}\Bigg].$$

Hence, taking into account (5.16), (5.17), (5.18) and using (4.4)-(4.6), after some calculations we obtain

$$||\ell||^2 = (-1)^{m+1}\Bigg[\frac{B}{(2m-1)!} - \frac{1}{(2m+1)!} - \sum_{j=0}^{m-3}\frac{B\ (-1)^j}{(2m-2-j)!\,(j+1)!} + \sum_{j=0}^{m-1}\frac{(-1)^j}{(j+1)!\,(2m-j)!} -$$

$$-\sum_{j=2}^{m-1}\frac{h^{2m-j}}{(2m-1-j)!(j+1)!}\left(\frac{B_{2m-j}}{2m-j} + \sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{-d_k q_k + p_k q_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i 0^{2m-1-j}\right) +$$

$$+\sum_{j=0}^{m-1}\frac{(-1)^{2m-1-j}}{(2m-1-j)!(j+1)!}\sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{2m-1-j} + \sum_{\beta=0}^{N} C[\gamma]\frac{(h\beta)^{2m}}{(2m)!}\Bigg]. \tag{5.36}$$

When $\alpha > m-1$ using lemma 5.1 and formulas (2.4)-(2.6) we get

$$\sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{\alpha} = \frac{1}{\alpha+1} + \sum_{j=1}^{\alpha-1}\frac{\alpha! B_{\alpha+1-j}}{j!(\alpha+1-j)!}h^{\alpha+1-j} +$$

$$+h^{\alpha+1}\sum_{k=1}^{m-1}\sum_{i=0}^{\alpha}\frac{d_k q_k^i + p_k q_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha} -$$

$$-\sum_{j=1}^{\alpha}\frac{\alpha! h^{j+1}}{j!(\alpha-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{d_k q_k^{N+i} + p_k q_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^j. \tag{5.37}$$

Using (5.37) and taking into account (2.1), (2.3), after simplications, from (5.36) we have

$$||\ell||^2 = (-1)^{m+1}\Bigg[\sum_{j=m}^{2m-3}\frac{B_{2m-j}h^{2m-j}}{(2m-j)!(j+1)!} + \frac{B_{2m}h^{2m}}{(2m)!} +$$

$$+\frac{h^{2m+1}}{(2m)!}\sum_{k=1}^{m-1}\sum_{i=0}^{2m}\frac{d_k q_k^i + p_k q_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^{2m} +$$

$$+\sum_{\alpha=m}^{2m-1}\frac{(-1)^{\alpha}}{(2m-\alpha)!}\sum_{j=1}^{\alpha-2}\frac{B_{\alpha+1-j}h^{\alpha+1-j}}{j!(\alpha+1-j)!} -$$

$$-\sum_{\alpha=m}^{2m}\frac{(-1)^{\alpha}}{(2m-\alpha)!}\sum_{j=2}^{\alpha}\frac{h^{j+1}}{j!(\alpha-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{d_k q_k^{N+i} + p_k q_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^j\Bigg].$$

Hence last two sums regrouping in powers of h, using designation (5.4*) and keepin in mind $Z_{\alpha-1} = \frac{B_\alpha}{\alpha}$, $\alpha = \overline{3, m}$, we have

$$\|\ell\|^2 = (-1)^{m+1} \left[\sum_{j=m}^{2m-3} \frac{B_{2m-j} h^{2m-j}}{(2m-j)!(j+1)!} + \frac{B_{2m} h^{2m}}{(2m)!} + \right.$$

$$+ \frac{h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} \sum_{i=0}^{2m} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1-q_k)^{i+1}} \Delta^i 0^{2m} +$$

$$+ \sum_{\alpha=3}^{m} \frac{B_\alpha h^\alpha}{\alpha!} \sum_{j=m}^{2m-1} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} + \sum_{\alpha=m+1}^{2m-1} \frac{B_\alpha h^\alpha}{\alpha!} \sum_{j=\alpha}^{2m-1} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} -$$

$$\left. - \sum_{\alpha=3}^{m} \frac{B_\alpha h^\alpha}{\alpha!} \sum_{j=m}^{2m} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} - \sum_{\alpha=m+1}^{2m+1} \frac{Z_{\alpha-1} h^\alpha}{(\alpha-1)!} \sum_{j=\alpha-1}^{2m} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} \right]. \tag{5.38}$$

Since in (5.38)

$$\sum_{j=\alpha-1}^{2m} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} = \frac{(-1)^{\alpha-1}}{(2m-\alpha-1)!} (1-1)^{2m-\alpha+1} = 0, \quad m+1 \le \alpha \le 2m$$

and

$$\sum_{j=\alpha}^{2m-1} \frac{(-1)^j}{(2m-j)!\,(j-\alpha+1)!} = -\frac{1}{(2m-\alpha+1)!} ((-1)^{\alpha-1} + 1),$$

then, using these equalities, from (5.38) we obtain

$$\|\ell\|^2 = (-1)^{m+1} \left[\frac{B_{2m} h^{2m}}{(2m)!} + \frac{h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} \sum_{i=0}^{2m} \frac{(1-q_k^N)(d_k q_k^i + (-1)^i p_k q_k)}{(1-q_k)^{i+1}} \Delta^i 0^{2m} - \right.$$

$$\left. - \sum_{j=m+1}^{2m-1} \frac{B_\alpha\ ((-1)^{\alpha-1} + 1)}{\alpha!\ (2m-\alpha+1)!} \right].$$

Hence taking into account that when α is even $(-1)^{\alpha-1} + 1 = 0$ and when α is odd $B_\alpha = 0$, (since $\alpha \neq 1$), we get the statement of theorem 5.2. Theorem 5.2 is proved.

Corollary 5.1. *In the space $L_2^{(2)}(0,1)$ among quadrature formulas of the form (1.1) with the error functional (1.2) there exists unique optimal formula which coefficients are determined by following formulas*

$$C[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$$

$$A = \frac{h^2}{12}, \quad B = -\frac{h^2}{12}.$$

Furthermore for square of norm of the error functional following is valid

$$\left\| \ell | L_2^{(2)*}(0,1) \right\|^2 = \frac{h^4}{720}.$$

Corollary 5.2. *In the space $L_2^{(3)}(0,1)$ among quadrature formulas of the form (1.1) with the error functional (1.2) there exists unique optimal formula which coefficients are determined by following formulas*

$$C[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$$

$$A = \frac{h^2}{12}, \quad B = -\frac{h^2}{12}.$$

Furthermore for square of norm of the error functional following is valid

$$\left\| \ell | L_2^{(3)*}(0,1) \right\|^2 = \frac{h^6}{30240}.$$

Proofs of Corollaries 5.1 and 5.2 we get immediately from theorems 5.1 and 5.2 when $m = 2$ and $m = 3$ respectively.

6 Acknowledgements

The second author gratefully acknowledges the Abdus Salam School of Mathematical Sciences (ASSMS), GC University, Lahore, Pakistan for providing the Post Doctoral Research Fellowship.

References

[1] A.Sard. Best approximate integration formulas, best approximate formulas, American J. of Math. V.71, No 1, (1949), pp.80-91.

[2] I.F.Meyers, A.Sard. Best approximate integration formulas, J. Math and Phys. XXIX, (1950), pp.118-123.

[3] I.J.Schoenberg and S.D.Silliman. On semicardinal quadrature formulae, Math. Comp., V.126, (1974), pp.483-497.

[4] F.Ya.Zagirova. On construction of optimal quadrature formulas with equal spaced nodes.-Novosibirsk, (1982), 28 p. (Preprint No 25, Institute of Mathematics SD of AS of USSR).

[5] S.L.Sobolev. The coefficients of optimal quadrature formulas, Selected Works of S.L.Sobolev. Springer, (2006). pp.561-566.

[6] Kh.M.Shadimetov. Optimal quadrature formulas in the $L_2^{(m)}(\Omega)$ and $L_2^{(m)}(R^1)$, Dokl. AN RUz, No 3, (1983), pp.5-8. (in Russian)

[7] Kh.M.Shadimetov. Construction of weight optimal quadrature formulas in the space $L_2^{(m)}(0, N)$, Siberian Journal of Computational Mathematics, V.5, No 3, (2002), pp.275-293. (in Russian)

[8] S.L.Sobolev. On the roots of Euler Polynomials. Selected Works of S.L.Sobolev. Springer, (2006). pp.567-572.

[9] Kh.M.Shadimetov. Optimal Formulas of Approximate Integration for differentiable Functions, Candidate dissertation. -Novosibirsk, (1983). 140p. (in Russian)

[10] R.W.Hamming. Numerical Methods for Scientists and Engenerees, NY, McGraw Bill Book Company, Inc. USA. 1962. 411p.

[11] A.O.Gelfond. Calculus of finite differences. - Moscow. Nauka, (1967). 376 p. (in Russian)

[12] S.L.Sobolev. Introduction to the Theory of Cubature Formulas. Moscow. Nauka, (1974) 808 p.

[13] S.L.Sobolev, V.L.Vaskevich. The Theory of Cubature Formulas, Kluwer Academic Publishers Group, Dordrecht, (1997). 416 p.

[14] Kh.M.Shadimetov. Discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}}$ and its construction, Problems of Computational and Applied Mathematics. -Tashkent, (1985). pp.22-35. (in Russian)

Shadimetov Kholmat Mahkambaevich

Institute of Mathematics and Information Technologies

Uzbek Academy of Sciences

29, Durmon yuli str

Tashkent, 100125, Uzbekistan

Hayotov Abdullo Rahmonovich

Institute of Mathematics and Information Technologies

Uzbek Academy of Sciences

29, Durmon yuli str

Tashkent, 100125, Uzbekistan

e-mail: hayotov@mail.ru, abdullo_hayotov@mail.ru

Nuraliev Farhod Abduganievich

Institute of Mathematics and Information Technologies

Uzbek Academy of Sciences

29, Durmon yuli str

Tashkent, 100125, Uzbekistan